

Stability of a non-uniform Rayleigh beam with general boundary conditions

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Abstract:

This paper investigates the stability of a Rayleigh beam with variable coefficients, clamped at one end and subject to general control laws at its free end. Based on the governing equations, we derive asymptotic expressions for the eigenvalues of the associated operator by means of a spectral method. Furthermore, we show that the optimal energy decay rate is determined by the spectral abscissa of the semigroup generator. We then show that exponential stability is maintained under a specific balance condition on the boundary control coefficients. Finally, numerical simulations are presented to illustrate the theoretical results.

Keywords : Rayleigh beam equation; Semigroup theory; Asymptotic expansions; Riesz basis; Exponential stability.

1. Introduction

The Rayleigh beam model extends the classical Euler-Bernoulli theory by incorporating the effects of rotatory inertia [1]. The stability analysis of this model has attracted considerable attention owing to its numerous applications in physical and engineering systems. In this study, we consider a Rayleigh beam with variable coefficients, clamped at one end and subject to

boundary controls at the free end. Here, the proposed control laws generalize the existing literature by considering both force and moment feedback simultaneously. The vibrations of the beam are governed by the following partial differential equation system; for $(x, t) \in (0, 1) \times (0, \infty)$:

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$$\begin{cases} r(x)y_{tt} - (I_r(x)y_{tt})_x + (EI(x)y_{xx})_{xx} = 0, \\ y(0, t) = y_x(0, t) = 0, \\ -EI(1)y_{xx}(1, t) = L_1(y_t, y_{xt}, y_x), \\ (EI(\cdot)y_{xx}(1, t))_x - I_r(1)y_{tt}(1, t) = L_2(y_t, y_{xt}, y), \\ y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \end{cases} \quad (1.1)$$

where:

- $y(x, t)$ denotes the transverse displacement at position x and time t ;
- $r(x)$ represents the mass density per unit length;
- $EI(x)$ is the flexural rigidity;
- $I_r(x)$ is the mass moment of inertia.

The coefficient functions $r(x)$, $I_r(x)$, and $EI(x)$ are assumed to be of class C^4 on the interval $[0, 1]$. Throughout this paper, subscripts denote partial derivatives with respect to time and space. We further assume the following positivity conditions for all $x \in [0, 1]$:

$$r(x) > 0, \quad I_r(x) > 0, \quad EI(x) > 0. \quad (1.2)$$

In addition, at the free end ($x = 1$), the beam is subject to a moment control L_1 and a force control L_2 . Specifically, L_1 is a linear combination of the velocity (y_t), angular velocity (y_{xt}), and rotation angle (y_x), while L_2 combines the velocity, rotational velocity, and displacement (y):

$$\begin{cases} L_1(y_t, y_{xt}, y_x) = (\kappa_{11}y_t + \kappa_{12}y_{xt} + \alpha y_x)(1, t), \\ L_2(y_t, y_{xt}, y) = (\kappa_{21}y_t + \kappa_{22}y_{xt} + \beta y)(1, t). \end{cases} \quad (1.3)$$

The control coefficients $\alpha, \beta, \kappa_{11}, \kappa_{22}, \kappa_{12}, \kappa_{21}$ are non-negative constants satisfying:

$$\begin{aligned} \alpha, \beta \geq 0, \quad \kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22} > 0 \\ \text{and} \quad 4\kappa_{12}\kappa_{21} > (\kappa_{11} + \kappa_{22})^2. \end{aligned} \quad (1.4)$$

Similar boundary conditions have been studied for the stability of Euler-Bernoulli beams in [2, 3]. However, the present work is the first to extend this approach to the Rayleigh case.

The total energy associated with system (1.1) is defined by:

$$\begin{aligned} E(t) = \frac{1}{2} \left(\int_0^1 r(x)|y_t|^2 dx + \int_0^1 I_r(x)|y_{tx}|^2 dx \right. \\ \left. + \int_0^1 EI(x)|y_{xx}|^2 dx \right. \\ \left. + \alpha|y_x(1, t)|^2 + \beta|y(1, t)|^2 \right). \end{aligned}$$

A straightforward calculation shows that the time derivative of the energy is:

$$\begin{aligned} \frac{d}{dt}E(t) = - \left(\kappa_{21}|y_t(1, t)|^2 + \kappa_{12}|y_{xt}(1, t)|^2 \right. \\ \left. + (\kappa_{22} + \kappa_{11})|y_t(1, t)y_{xt}(1, t)| \right) \leq 0. \end{aligned}$$

Under the condition (1.4), the quadratic form in the expression above is positive, ensuring that $\frac{d}{dt}E(t) \leq 0$.

Consequently, the energy $E(t)$ is non-increasing.

The stabilization of system (1.1) has been investigated using various types of control laws. In [4–6], the authors considered the laws $L_1(y_{xt}) = \kappa_{12}y_{xt}(1, t)$ and $L_2(y_t) = \kappa_{21}y_t(1, t)$, addressing both uniform and non-uniform Rayleigh beam problems. In both cases, they established the Riesz basis property and the exponential stability of the system. We recall that the Riesz basis property implies that the generalized eigenvectors of the system constitute an unconditional basis for the Hilbert space. Other studies have focused on the perturbed version of system (1.1). In [7], Wang and Yung examined a damped case with zero boundary controls, proving that the system is exponentially stable under certain conditions. Furthermore, Kouassi *et al.* [8] recently investigated the perturbed system (1.1) through the control laws $L_1(y_{xt}, y_x) = (\kappa_{12}y_{xt} + \alpha y_x)(1, t)$ and $L_2(y_t, y) = (\kappa_{21}y_t + \beta y)(1, t)$. These authors derived the asymptotic expressions for the eigenvalues and established both the Riesz basis property and exponential stability.

In contrast to [5–8], which focus solely on decoupled control laws for either rotational or linear velocity at the free end, this study introduces a full family of coupled boundary controls (1.3) that simultaneously include velocity and rotational velocity. To analyze system (1.1) under these general boundary controls, we impose conditions (1.4) to ensure energy dissipativity.

The primary objective of this work is to demonstrate that the system remains exponentially stable under the boundary controls (1.3). The novelty of our approach lies in the generalization of several existing results through a judicious selection of control parameters. This generalization leads to new spectral challenges, particularly in the analysis of the characteristic determinant and the precise characterization of exponential stability. To address the technical challenges encountered, we employ a spectral method and integrate the techniques developed in [9, 10]. Finally, we establish the Riesz basis property and prove that the system is exponentially stable if $\kappa_{11} = \kappa_{22}$ holds.

The paper is organized as follows. After presenting the well-posedness in Section 2., we perform a spectral analysis of the system operator in Section 3.. Section 4. is dedicated to establishing the Riesz basis property and demonstrating the stability of the system. In Section 5., using the finite element method, we present numerical simulations that enable us to study the spectrum of the system and the effect of the control parameters on the system's stability. We provide a comparative discussion in Section 6. and some concluding remarks in Section 7..

2. Well-posedness

In this section, we establish the well-posedness of problem (1.1). To this end, we introduce the following Hilbert spaces:

$$H_e^1(0, 1) = \{u \in H^1(0, 1) : u(0) = 0\},$$

$$\|u\|_{H_e^1}^2 = \int_0^1 (r(x)|u|^2 + I_r(x)|u_x|^2) dx, \text{ and}$$

$$H_e^2(0, 1) = \{u \in H^2(0, 1) : u(0) = u_x(0) = 0\},$$

$$\|u\|_{H_e^2}^2 = \int_0^1 EI(x)|u_{xx}|^2 dx + \alpha|u_x(1)|^2 + \beta|u(1)|^2.$$

We consider the energy space: $\mathbb{H} = H_e^2 \times H_e^1$, equipped with the inner product defined for $U = (u_1, u_2) \in \mathbb{H}$ and $V = (v_1, v_2) \in \mathbb{H}$ by:

$$\begin{aligned} \langle U, V \rangle_{\mathbb{H}} &= \int_0^1 r(x)u_2(x)\overline{v_2(x)}dx \\ &+ \int_0^1 I_r(x)(u_2)_x(x)\overline{(v_2)_x(x)}dx \\ &+ \int_0^1 EI(x)(u_1)_{xx}(x)\overline{(v_1)_{xx}(x)}dx \\ &+ \alpha(u_1)_x(1)\overline{(v_1)_x(1)} + \beta u_1(1)\overline{v_1(1)}. \end{aligned}$$

Identifying $L^2(0, 1)$ with its dual, we have the following continuous embedding:

$$H_e^2 \subset H_e^1 \subset L^2(0, 1) \subset H_e^{-1} \subset H_e^{-2},$$

where H_e^{-1} and H_e^{-2} are the dual spaces of H_e^1 and H_e^2 , respectively.

Now, let y be a smooth solution of system (1.1). Then multiplying the first equation of (1.1) by a function $\bar{\phi} \in H_e^2$ and integrating by parts over $[0, 1]$, we get:

$$\begin{aligned} &\int_0^1 r(x)y_{tt}(x)\bar{\phi}(x)dx + \int_0^1 I_r(x)y_{ttx}(x)\bar{\phi}_x(x)dx \\ &- \left[I_r(x)y_{ttx}(x)\bar{\phi}(x) \right]_0^1 + \int_0^1 EI(x)y_{xx}(x)\bar{\phi}_{xx}(x)dx \\ &+ \left[(EI(x)y_{xx}(x))_x\bar{\phi}(x) \right]_0^1 - \left[EI(x)y_{xx}(x)\bar{\phi}_x(x) \right]_0^1 \\ &= 0. \end{aligned}$$

By substituting the boundary conditions from (1.1) into the terms above, we obtain:

$$\begin{aligned} &\int_0^1 r(x)y_{tt}(x)\bar{\phi}(x)dx + \int_0^1 I_r(x)y_{ttx}(x)\bar{\phi}_x(x)dx \\ &+ \int_0^1 EI(x)y_{xx}(x)\bar{\phi}_{xx}(x)dx + \kappa_{21}y_t(1)\bar{\phi}(1) \\ &+ \kappa_{22}y_{xt}(1)\bar{\phi}(1) + \beta y(1)\bar{\phi}(1) + \kappa_{11}y_t(1)\bar{\phi}_x(1) \\ &+ \kappa_{12}y_{xt}(1)\bar{\phi}_x(1) + \alpha y_x(1)\bar{\phi}_x(1) = 0. \end{aligned} \quad (2.1)$$

To facilitate the analysis, we define the linear operators $\hat{A}, \hat{F} \in \mathcal{L}(H_e^2, H_e^{-2})$ and $\hat{B}, \hat{C}, \hat{D}, \hat{E} \in \mathcal{L}(H_e^1, H_e^{-1})$ as follows:

$$\begin{aligned} \langle \hat{A}\phi, \psi \rangle &= \int_0^1 EI(x)\phi_{xx}(x)\overline{\psi_{xx}(x)}dx \\ &+ \beta\phi(1)\overline{\psi(1)} + \alpha\phi_x(1)\overline{\psi_x(1)}, \quad \forall \phi, \psi \in H_e^2, \\ \langle \hat{C}\phi, \psi \rangle &= \int_0^1 r(x)\phi(x)\overline{\psi(x)}dx \\ &+ \int_0^1 I_r(x)\phi_x(x)\overline{\psi_x(x)}dx, \quad \forall \phi, \psi \in H_e^1, \end{aligned}$$

$$\begin{aligned} \langle \hat{B}\phi, \psi \rangle &= \phi(1)\overline{\psi(1)}, \quad \forall \phi, \psi \in H_e^1, \\ \langle \hat{D}\phi, \psi \rangle &= \phi_x(1)\overline{\psi(1)}, \quad \forall \phi, \psi \in H_e^1, \\ \langle \hat{E}\phi, \psi \rangle &= \phi(1)\overline{\psi_x(1)}, \quad \forall \phi, \psi \in H_e^1, \\ \langle \hat{F}\phi, \psi \rangle &= \phi_x(1)\overline{\psi_x(1)}, \quad \forall \phi, \psi \in H_e^2. \end{aligned} \quad (2.2)$$

By means of the Lax-Milgram theorem [11], we see that $\hat{A}$ (resp. $\hat{C}$) is the canonical isomorphism from H_e^2 onto H_e^{-2} and from H_e^1 onto H_e^{-1} respectively. With these operators, the system (1.1) can be written as a variational equation:

$$\begin{aligned} \langle \hat{C}y_{tt}, \psi \rangle + \langle \hat{A}y, \psi \rangle + \kappa_{21}\langle \hat{B}y_t, \psi \rangle \\ + \kappa_{22}\langle \hat{D}y_t, \psi \rangle + \kappa_{11}\langle \hat{E}y_t, \psi \rangle \\ + \kappa_{12}\langle \hat{F}y_t, \psi \rangle = 0 \end{aligned} \quad (2.3)$$

for $\psi \in H_e^2 \cap H^4$. More simply, we have:

$$\langle \hat{C}y_{tt}, \psi \rangle + \langle \hat{A}y, \psi \rangle + \langle \hat{A}_0y_t, \psi \rangle = 0,$$

where $\hat{A}_0$ is the operator defined by:

$$\hat{A}_0 = \kappa_{21}\hat{B} + \kappa_{22}\hat{D} + \kappa_{11}\hat{E} + \kappa_{12}\hat{F}.$$

Next, to characterize the evolution system, we introduce the linear unbounded operator $\mathcal{A}$, which is defined on its domain $D(\mathcal{A})$ by:

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{H} : v \in H_e^2, \right. \\ \left. \hat{A}u + \hat{A}_0v \in H_e^{-1} \right\}, \quad (2.4)$$

and acting as follows:

$$\mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{pmatrix} v \\ -\hat{C}^{-1}(\hat{A}u + \hat{A}_0v) \end{pmatrix}. \quad (2.5)$$

By setting $Y(t) = (y, y_t)$, the system (2.3) can be rewritten in the evolution form:

$$\begin{cases} \frac{dY(t)}{dt} = \mathcal{A}Y(t), & t > 0, \\ Y(0) = Y_0 \in \mathbb{H}. \end{cases} \quad (2.6)$$

Before stating the main result of this section, we present two preliminary lemmas that will be used throughout this paper.

Lemma 2.1. *Let $\phi \in \mathbb{C}$ and $\psi \in \mathbb{C}$, we have the following equalities:*

$$\begin{aligned} \operatorname{Re}(\psi(1)\bar{\phi}(1)) &= \operatorname{Re}(\overline{\psi(1)}\phi(1)), \\ \operatorname{Im}(\psi(1)\bar{\phi}(1)) &= -\operatorname{Im}(\overline{\psi(1)}\phi(1)), \\ \operatorname{Re}(\psi(1)\bar{\phi}(1)) &= \operatorname{Re}\psi(1) \times \operatorname{Re}\bar{\phi}(1) \\ &\quad - \operatorname{Im}\psi(1) \times \operatorname{Im}\bar{\phi}(1), \\ \operatorname{Im}(\psi(1)\bar{\phi}(1)) &= \operatorname{Re}\psi(1) \times \operatorname{Im}\bar{\phi}(1) \\ &\quad + \operatorname{Im}\psi(1) \times \operatorname{Re}\bar{\phi}(1) \end{aligned}$$

where $\overline{\phi(1)}$ is the conjugate of $\phi(1)$.

Proof. The results follow by decomposing $\psi(1)$ and $\overline{\phi(1)}$ into their real and imaginary parts: $\psi(1) = \mathcal{R}e\psi(1) + i\mathcal{I}m\psi(1)$ and $\overline{\phi(1)} = \mathcal{R}e\overline{\phi(1)} + i\mathcal{I}m\overline{\phi(1)}$.

Lemma 2.2. Let $\phi, \psi \in \mathbb{C}$. Assume that (1.4) holds and $(\phi(1), \psi(1)) \neq (0, 0)$. Then:

$$\kappa_{21}|\phi(1)|^2 + \kappa_{12}|\psi(1)|^2 + (\kappa_{11} + \kappa_{22})\mathcal{R}e(\psi(1)\overline{\phi(1)}) > 0. \quad (2.7)$$

Proof. We distinguish two cases based on the sign of $\mathcal{R}e(\psi(1)\overline{\phi(1)})$.

Case 1: Assume that $\mathcal{R}e(\psi(1)\overline{\phi(1)}) \geq 0$.

Since $\kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22} > 0$, the inequality (2.7) holds trivially.

Case 2: Assume that $\mathcal{R}e(\psi(1)\overline{\phi(1)}) < 0$. From (1.4) we have:

$$2\kappa_{12}^{\frac{1}{2}}\kappa_{21}^{\frac{1}{2}} > (\kappa_{11} + \kappa_{22}). \quad (2.8)$$

Multiplying (2.8) by $\mathcal{R}e(\psi(1)\overline{\phi(1)})$ we have:

$$2\kappa_{12}^{\frac{1}{2}}\kappa_{21}^{\frac{1}{2}}\mathcal{R}e(\psi(1)\overline{\phi(1)}) < (\kappa_{11} + \kappa_{22})\mathcal{R}e(\psi(1)\overline{\phi(1)}).$$

It follows that:

$$\begin{aligned} & \kappa_{21}|\phi(1)|^2 + 2\kappa_{12}^{\frac{1}{2}}\kappa_{21}^{\frac{1}{2}}\mathcal{R}e(\psi(1)\overline{\phi(1)}) + \kappa_{12}|\psi(1)|^2 \\ & < \kappa_{21}|\phi(1)|^2 + \kappa_{12}|\psi(1)|^2 + (\kappa_{11} + \kappa_{22})\mathcal{R}e(\psi(1)\overline{\phi(1)}). \end{aligned}$$

By expanding the left-hand side into real and imaginary parts, we obtain:

$$\begin{aligned} & \kappa_{21}|\phi(1)|^2 + 2\kappa_{12}^{\frac{1}{2}}\kappa_{21}^{\frac{1}{2}}\mathcal{R}e(\psi(1)\overline{\phi(1)}) + \kappa_{12}|\psi(1)|^2 \\ & = \kappa_{21}[\mathcal{R}e\phi(1)]^2 + 2\kappa_{12}^{\frac{1}{2}}\kappa_{21}^{\frac{1}{2}}[\mathcal{R}e\psi(1)\mathcal{R}e\phi(1)] \\ & \quad + \kappa_{12}[\mathcal{R}e\psi(1)]^2 + \kappa_{21}[\mathcal{I}m\phi(1)]^2 \\ & \quad + 2\kappa_{12}^{\frac{1}{2}}\kappa_{21}^{\frac{1}{2}}[\mathcal{I}m\psi(1)\mathcal{I}m\phi(1)] + \kappa_{12}[\mathcal{I}m\psi(1)]^2 \\ & = \left[\kappa_{21}^{\frac{1}{2}}\mathcal{R}e\phi(1) + \kappa_{12}^{\frac{1}{2}}\mathcal{R}e\psi(1)\right]^2 \\ & \quad + \left[\kappa_{21}^{\frac{1}{2}}\mathcal{I}m\phi(1) + \kappa_{12}^{\frac{1}{2}}\mathcal{I}m\psi(1)\right]^2. \end{aligned}$$

Since this sum of squares is non-negative (and strictly positive for $(\phi(1), \psi(1)) \neq (0, 0)$), the proof is complete.

We now establish the main result of this section. Specifically, we characterize the properties of the operator $\mathcal{A}$ as follows.

Theorem 2.1. Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \rightarrow \mathbb{H}$ be defined by (2.4)–(2.5). Then $\mathcal{A}$ is a densely defined closed dissipative operator with $0 \in \rho(\mathcal{A})$, and consequently, $\mathcal{A}$ generates a C_0 -semigroup of contractions.

Proof. First, we show that $\mathcal{A}$ is dissipative. For any $U = (y, y_t)^T \in D(\mathcal{A})$ we have:

$$\langle \mathcal{A}U, U \rangle_{\mathbb{H}} = \left\langle \begin{pmatrix} y_t \\ -\widehat{C}^{-1}(\widehat{A}y + \widehat{A}_0y_t) \end{pmatrix}, \begin{pmatrix} y \\ y_t \end{pmatrix} \right\rangle_{\mathbb{H}}.$$

Expanding $\langle \mathcal{A}U, U \rangle_{\mathbb{H}}$ yields:

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathbb{H}} &= \int_0^1 EI(x)y_{txx}(x)\overline{y_{xx}(x)}dx + \alpha y_{xt}(1)\overline{y_x(1)} \\ & \quad + \beta y_t(1)\overline{y(1)} - \int_0^1 r(x)\widehat{C}^{-1}(\widehat{A}y + \widehat{A}_0y_t)\overline{y_t}dx \\ & \quad - \int_0^1 I_r(x)\widehat{C}^{-1}(\widehat{A}y + \widehat{A}_0y_t)\overline{y_{tx}}dx \\ &= \langle \widehat{A}y_t, y \rangle - \left\langle \widehat{C}^{-1}(\widehat{A}y + \widehat{A}_0y_t), y_t \right\rangle \\ &= \langle \widehat{A}y_t, y \rangle - \langle \widehat{A}y + \widehat{A}_0y_t, y_t \rangle \\ &= \langle \widehat{A}y_t, y \rangle - \langle \widehat{A}y, y_t \rangle - \langle \widehat{A}_0y_t, y_t \rangle. \end{aligned}$$

Since

$$\begin{aligned} \langle \widehat{A}_0y_t, y_t \rangle &= \kappa_{21}\langle \widehat{B}y_t, y_t \rangle + \kappa_{22}\langle \widehat{D}y_t, y_t \rangle + \kappa_{11}\langle \widehat{E}y_t, y_t \rangle \\ & \quad + \kappa_{12}\langle \widehat{F}y_t, y_t \rangle, \end{aligned}$$

therefore:

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathbb{H}} &= \langle \widehat{A}y_t, y \rangle - \overline{\langle \widehat{A}y_t, y \rangle} - \kappa_{21}|y_t(1)|^2 - \kappa_{11}y_t(1)\overline{y_{xt}(1)} \\ & \quad - \kappa_{12}|y_{xt}(1)|^2 - \kappa_{22}y_{xt}(1)\overline{y_t(1)}. \end{aligned}$$

From above and (2.7) we conclude that:

$$\begin{aligned} \mathcal{R}e\langle \mathcal{A}U, U \rangle_{\mathbb{H}} &= -\left(\kappa_{21}|y_t(1)|^2 + \kappa_{12}|y_{xt}(1)|^2 \right. \\ & \quad \left. + (\kappa_{11} + \kappa_{22})\mathcal{R}e(y_{xt}(1)\overline{y_t(1)})\right) \leq 0, \end{aligned}$$

which proves that $\mathcal{A}$ is dissipative.

Next, we show that $0 \in \rho(\mathcal{A})$. We look for $(f, g)^T \in \mathbb{H}$ such that:

$$\mathcal{A}(u, v)^T = (f, g)^T,$$

which implies that:

$$f = v, \quad g = -\widehat{C}^{-1}(\widehat{A}u + \widehat{A}_0v).$$

Therefore, for any $\varphi \in H_e^2$,

$$\langle \widehat{A}u + \widehat{A}_0v, \varphi \rangle = -\langle \widehat{C}g, \varphi \rangle.$$

Substituting $v = f$ into above equation yields:

$$\langle \widehat{A}u, \varphi \rangle = -\langle \widehat{A}_0f, \varphi \rangle - \langle \widehat{C}g, \varphi \rangle.$$

In variational form, this is equivalent to finding $u \in H_e^2$ such that:

$$a(u, \varphi) = L(\varphi), \quad \varphi \in H_e^2, \quad (2.9)$$

where the bilinear form $a(\cdot, \cdot)$ and the linear form $L(\cdot)$ are defined by:

$$\begin{aligned} a(u, \varphi) &= \langle \widehat{A}u, \varphi \rangle, \\ L(\varphi) &= -\langle \widehat{A}_0f, \varphi \rangle - \langle \widehat{C}g, \varphi \rangle \\ &= -\langle \widehat{C}g + \kappa_{21}\widehat{B}f + \kappa_{22}\widehat{D}f + \kappa_{11}\widehat{E}f + \kappa_{12}\widehat{F}f, \varphi \rangle. \end{aligned}$$

For any $\varphi \in H_e^2$, we have:

$$a(\varphi, \varphi) = \|\varphi\|_{H_e^2}^2.$$

Since $a(\varphi, \varphi) = \|\varphi\|_{H_e^2}^2$, the form $a(\cdot, \cdot)$ is continuous and coercive on H_e^2 . Furthermore, $L(\cdot)$ is a continuous linear functional on H_e^2 . By the Lax-Milgram theorem, there exists a unique $u \in H_e^2$ satisfying (2.9). Thus, $0 \in \rho(\mathcal{A})$. The rest of the theorem follows from the general theory of C_0 -semigroups (see [12]).

In addition, we have the following characterization of the space $D(\mathcal{A})$ (see [13]).

Lemma 2.3. *Let $(u, v)^T \in \mathbb{H}$. Then $(u, v)^T \in D(\mathcal{A})$ if and only if $u \in H^3 \cap H_e^2$ and $v \in H_e^2$ such that the following boundary conditions hold:*

$$\begin{aligned} & \left(EI(z)u''(z) \right)' \Big|_{z=1} - \beta u(1) - \kappa_{21}v(1) \\ & - \kappa_{22}v'(1) + I_r(z) \left(\widehat{C}^{-1} \left[\widehat{A}u + \kappa_{21}\widehat{B}v \right. \right. \\ & \left. \left. + \kappa_{22}\widehat{D}v + \kappa_{11}\widehat{E}v + \kappa_{12}\widehat{F}v \right] \right)' \Big|_{z=1} = 0, \end{aligned}$$

and

$$EI(1)u''(1) + \alpha u'(1) + \kappa_{11}v(1) + \kappa_{12}v'(1) = 0.$$

Then, $\mathcal{A}^{-1}$ is a compact operator on $\mathbb{H}$.

Proof. The sufficiency is obvious. To prove the necessity, let $(u, v)^T \in D(\mathcal{A})$ be such that

$$\mathcal{A}(u, v)^T = (f, g)^T \in \mathbb{H}.$$

This leads to:

$$f = v \quad \text{and} \quad g = -\widehat{C}^{-1}(\widehat{A}u + \widehat{A}_0v).$$

Since $g \in H_e^1$ and $\widehat{C} : H_e^1 \rightarrow H_e^{-1}$ is an isomorphism, it follows that:

$$\widehat{A}u + \widehat{A}_0f = -\widehat{C}g \quad \text{in} \quad H_e^{-1} \subset H_e^{-2}.$$

Thus, for all $\psi \in H_e^2$, we have:

$$\begin{aligned} & \int_0^1 EI(z)u''(z)\overline{\psi''(z)}dz + \beta u(1)\overline{\psi(1)} \\ & + \int_0^1 r(z)g(z)\overline{\psi(z)}dz + \int_0^1 I_r(z)g'(z)\overline{\psi'(z)}dz \quad (2.10) \\ & + \alpha u'(1)\overline{\psi'(1)} + \kappa_{21}f(1)\overline{\psi(1)} + \kappa_{22}f'(1)\overline{\psi'(1)} \\ & + \kappa_{11}f(1)\overline{\psi'(1)} + \kappa_{12}f'(1)\overline{\psi'(1)} = 0. \end{aligned}$$

For any $\phi \in C_0^\infty(0, 1)$, let $\psi(z) = \int_0^z \phi(s)ds$ with $\psi(1) = 0$. Substituting ψ into (2.10), we obtain:

$$\begin{aligned} & \int_0^1 EI(z)u''(z)\overline{\phi'(z)}dz + \beta u(1)\int_0^1 \overline{\phi(z)}dz \\ & + \int_0^1 \overline{\phi(z)}dz \int_z^1 r(s)g(s)ds \\ & + \int_0^1 I_r(z)g'(z)\overline{\phi(z)}dz + \kappa_{21}f(1)\int_0^1 \overline{\phi(z)}dz \\ & + \kappa_{22}f'(1)\int_0^1 \overline{\phi(z)}dz = 0. \end{aligned}$$

Differentiating the first term, we deduce that:

$$\begin{aligned} & \int_0^1 \left(EI(z)u''(z) \right)' \overline{\phi(z)}dz \\ & = \int_0^1 \left[\beta u(1) + \kappa_{21}f(1) + \kappa_{22}f'(1) \right. \\ & \left. + \int_z^1 r(s)g(s)ds + I_r(z)g'(z) \right] \overline{\phi(z)}dz, \end{aligned}$$

for all $\phi \in H_e^2$. Therefore,

$$\begin{aligned} \left(EI(z)u''(z) \right)' & = \beta u(1) + \kappa_{21}f(1) + \kappa_{22}f'(1) \\ & + \int_z^1 r(s)g(s)ds + I_r(z)g'(z). \end{aligned} \quad (2.11)$$

Since $\left(EI(z)u''(z) \right)' \in L^2(0, 1)$ and $EI \in C^4([0, 1])$, it follows that $u \in H^3 \cap H_e^2$.

In particular, by evaluating (2.11) at $z = 1$, we get:

$$\begin{aligned} \left(EI(z)u''(z) \right)' \Big|_{z=1} & = \beta u(1) + \kappa_{21}f(1) + \kappa_{22}f'(1) \\ & + I_r(z)g'(z) \Big|_{z=1} \end{aligned} \quad (2.12)$$

Substituting $f = v$ and the definition of g into (2.12) yields the first boundary condition:

$$\begin{aligned} & \left(EI(z)u''(z) \right)' \Big|_{z=1} - \beta u(1) - \kappa_{21}v(1) \\ & - \kappa_{22}v'(1) + I_r(z) \left(C^{-1} \left[\widehat{A}u + \widehat{A}_0f \right] \right)' \Big|_{z=1} = 0. \end{aligned}$$

Similarly, by choosing $\phi \in H_e^1$ such that $\phi(1) = 1$ and defining $\psi(z) = \int_0^z \phi(s)ds$, we substitute this into (2.10) to obtain:

$$\begin{aligned} & \int_0^1 EI(z)u''(z)\overline{\phi'(z)}dz + \alpha u'(1) + \kappa_{11}f(1) + \kappa_{12}f'(1) \\ & + \int_0^1 \left[\beta u(1) + \kappa_{21}f(1) + \kappa_{22}f'(1) \right. \\ & \left. + \int_z^1 r(s)g(s)ds + I_r(z)g'(z) \right] \overline{\phi(z)}dz = 0. \end{aligned}$$

Using (2.11) and $f = v$, we have:

$$\int_0^1 \left(EI(z)u''(z)\overline{\phi(z)} \right)' dz + \alpha u'(1) + \kappa_{11}f(1) + \kappa_{12}f'(1) = 0.$$

Integrating the above equation by parts, we obtain:

$$EI(1)u''(1) + \alpha u'(1) + \kappa_{11}v(1) + \kappa_{12}v'(1) = 0.$$

which is the second boundary condition.

Finally, since $\mathcal{A}^{-1}$ exists, the Sobolev Embedding Theorem ensures that $\mathcal{A}^{-1}$ is a compact operator on $\mathbb{H}$. Consequently, the spectrum of $\mathcal{A}$ consists essentially of isolated eigenvalues of finite multiplicity, distributed in the complex plane as conjugate pairs.

3. Spectral analysis of the operator $\mathcal{A}$

In this section, we investigate the eigenvalue problem associated with the operator $\mathcal{A}$ to characterize the behavior of its spectrum. Following the approach developed in [6], let $\lambda \in \sigma(\mathcal{A})$ be an eigenvalue and $(\phi, \lambda\phi)^T \in D(\mathcal{A})$ be the corresponding eigenfunction. Then, ϕ satisfies the following boundary value problem for $x \in [0, 1]$:

$$\begin{cases} \lambda^2 r(x)\phi(x) - \lambda^2 (I_r(x)\phi'(x))' + (EI(x)\phi''(x))'' = 0, \\ \phi(0) = \phi'(0) = 0, \\ -EI(1)\phi''(1) = \kappa_{11}\lambda\phi(1) + (\kappa_{12}\lambda + \alpha)\phi'(1), \\ (EI(\cdot)\phi'')'(1) - \lambda^2 I_r(1)\phi'(1) = \kappa_{22}\lambda\phi'(1) \\ \quad + (\kappa_{21}\lambda + \beta)\phi(1). \end{cases} \quad (3.1)$$

Expanding the derivatives in (3.1), we obtain:

$$\begin{cases} \phi^{(4)}(x) + \frac{2EI'(x)}{EI(x)}\phi'''(x) + \frac{EI''(x)}{EI(x)}\phi''(x) \\ \quad - \lambda^2 \left(\frac{I_r(x)}{EI(x)}\phi''(x) + \frac{I_r'(x)}{EI(x)}\phi'(x) \right. \\ \quad \left. - \frac{r(x)}{EI(x)}\phi(x) \right) = 0, \\ \phi(0) = \phi'(0) = 0, \\ EI(1)\phi''(1) + \kappa_{12}\lambda\phi'(1) + \alpha\phi'(1) \\ \quad + \kappa_{11}\lambda\phi(1) = 0, \\ EI(1)\phi'''(1) + EI'(1)\phi''(1) - \lambda^2 I_r(1)\phi'(1) \\ \quad - \kappa_{22}\lambda\phi'(1) - \kappa_{21}\lambda\phi(1) - \beta\phi(1) = 0. \end{cases} \quad (3.2)$$

Following the approach in [5–7], we transform the coefficient $\frac{I_r(x)}{EI(x)}$ of ϕ'' into a uniform form by introducing a spatial scale transformation:

$$\begin{aligned} f(z) &:= \phi(x), \quad \text{and} \\ z &:= \frac{1}{h} \int_0^x \left(\frac{I_r(\xi)}{EI(\xi)} \right)^{\frac{1}{2}} d\xi, \end{aligned} \quad (3.3)$$

where

$$h = \int_0^1 \left(\frac{I_r(\xi)}{EI(\xi)} \right)^{\frac{1}{2}} d\xi.$$

Under this transformation, a straightforward derivation shows that ϕ satisfies:

$$\begin{aligned} \phi'(x) &= z_x(x)f'(z(x)) \\ \phi''(x) &= z_{xx}(x)f'(z(x)) + z_x^2(x)f''(z(x)) \\ \phi'''(x) &= z_{xxx}(x)f'(z(x)) + z_x^3(x)f'''(z(x)) \\ &\quad + 3z_x(x)z_{xx}(x)f''(z(x)) \\ \phi^{(4)}(x) &= z_{xxxx}(x)f'(z(x)) + z_x^4(x)f^{(4)}(z(x)) \\ &\quad + (3z_{xx}^2(x) + 4z_x(x)z_{xxx}(x))f''(z(x)) \\ &\quad + 6z_{xx}(x)z_x^2(x)f'''(z(x)). \end{aligned}$$

After substitution, f satisfies the following system for $z \in [0, 1]$:

$$\begin{cases} f^{(4)}(z) + a_1(z)f'''(z) + a_2(z)f''(z) \\ \quad + a_3(z)f'(z) - h^2\lambda^2 \left[f''(z) \right. \\ \quad \left. + b_1(z)f'(z) - b_2(z)f(z) \right] = 0, \\ f(0) = f'(0) = 0, \\ c_{21}f''(1) + c_{22}f'(1) + c_{23}\kappa_{12}\lambda f'(1) \\ \quad + c_{23}\alpha f'(1) + \kappa_{11}\lambda f(1) = 0, \\ c_{11}f'''(1) + c_{12}f''(1) + c_{13}f'(1) \\ \quad - \lambda^2 c_{14}f'(1) - \kappa_{22}c_{15}\lambda f'(1) \\ \quad - \kappa_{21}\lambda f(1) - \beta f(1) = 0, \end{cases} \quad (3.4)$$

where

$$\begin{aligned} a_1(z) &= \frac{6z_{xx}}{z_x^2} + \frac{2EI'(x)}{z_x EI(x)}, \\ a_2(z) &= \frac{3z_{xx}^2}{z_x^4} + \frac{4z_{xxx}}{z_x^3} + \frac{6z_{xx}EI'(x)}{z_x^3 EI(x)} + \frac{EI''(x)}{z_x^2 EI(x)}, \\ a_3(z) &= \frac{z_{xxxx}}{z_x^4} + \frac{2z_{xxx}EI'(x)}{z_x^4 EI(x)} + \frac{z_{xx}EI''(x)}{z_x^4 EI(x)}, \\ b_1(z) &= \frac{z_{xx}}{z_x^2} + \frac{I_r'(x)}{h^2 z_x^3 EI(x)} = \frac{z_{xx}}{z_x^2} + \frac{I_r'(x)}{z_x I_r(x)}, \\ b_2(z) &= \frac{r(x)}{h^2 z_x^4 EI(x)} = \frac{r(x)}{z_x^2 I_r(x)}, \end{aligned}$$

$$\begin{aligned} c_{11} &= z_x^3(1)EI(1), \quad c_{14} = I_r(1)z_x(1) \\ c_{12} &= 3z_x(1)z_{xx}(1)EI(1) + z_x^2 EI'(1), \\ c_{13} &= z_{xxx}(1)EI(1) + z_{xx}(1)EI'(1), \\ c_{15} &= z_x(1), \quad c_{21} = z_x^2(1)EI(1), \\ c_{22} &= z_{xx}(1)EI(1), \quad c_{23} = z_x(1), \end{aligned}$$

$$\begin{aligned} z_x &= \frac{1}{h} \left(\frac{I_r(x)}{EI(x)} \right)^{\frac{1}{2}}, \quad z_x^2 = \frac{I_r(x)}{h^2 EI(x)} \\ z_{xx} &= \frac{1}{2h} \left(\frac{I_r(x)}{EI(x)} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{I_r(x)}{EI(x)} \right)^{\frac{1}{2}}. \end{aligned}$$

By substituting $\mu = \lambda h$ into (3.4), we obtain the following eigenvalue problem:

$$\begin{cases} f^{(4)}(z) + a_1(z)f'''(z) + a_2(z)f''(z) + a_3(z)f'(z) \\ \quad - \mu^2 \left[f''(z) + b_1(z)f'(z) - b_2(z)f(z) \right] = 0, \\ B_4(f) = f(0) = 0, \\ B_3(f) = f'(0) = 0, \\ B_2(f) = f''(1) + \frac{c_{22}}{c_{21}}f'(1) \\ \quad + \frac{c_{23}}{c_{21}}\kappa_{12}\mu h^{-1}f'(1) + \frac{c_{23}}{c_{21}}\alpha f'(1) \\ \quad + \frac{1}{c_{21}}\kappa_{11}\mu h^{-1}f(1) = 0, \\ B_1(f) = f'''(1) + \frac{c_{12}}{c_{11}}f''(1) + \frac{c_{13}}{c_{11}}f'(1) \\ \quad - \frac{c_{14}}{c_{11}}\mu^2 h^{-2}f'(1) - \kappa_{22}\frac{c_{15}}{c_{11}}\mu h^{-1}f'(1) \\ \quad - \frac{1}{c_{11}}\kappa_{21}\mu h^{-1}f(1) - \frac{\beta}{c_{11}}f(1) = 0, \end{cases} \quad (3.5)$$

which is equivalent to the system (3.2).

Remark 3.1. The spectrum $\sigma(\mathcal{A})$ is characterized by the existence of a non-trivial solution $f(z)$ to (3.5) for $\mu = \lambda h$, where the function $\phi(x)$ in the corresponding eigenfunction $(\phi, \lambda\phi)$ of $\mathcal{A}$ is defined by (3.3).

To analyze problem (3.5), we require the following lemma (see [1], Lemma 8.2.3, for a detailed proof).

Lemma 3.1. Let $h_1(x)$ and $h_2(x)$ be two linearly independent solutions to the second-order linear homogeneous differential equation:

$$(I_r(x)\phi'(x))' - r(x)\phi(x) = 0. \quad (3.6)$$

Then, there exists a constant Γ such that:

$$\Gamma = h_2'(0)h_1(1) - h_1'(1)h_2(0) \neq 0. \quad (3.7)$$

We are now in a position to study the asymptotic behavior of the eigenvalues of $\mathcal{A}$. To this end, we determine the fundamental solutions of (3.5) and use them to expand the characteristic determinant of $\mathcal{A}$. Following the method described in [14], we divide the complex plane $\mathbb{C}$ into four sectors:

$$\mathcal{S}_k := \left\{ z \in \mathbb{C} : \frac{k\pi}{2} \leq \arg(z) \leq \frac{(k+1)\pi}{2} \right\}, \\ k = 0, 1, \dots, 3.$$

Next, let ω_1 and ω_2 be the two square roots of -1 , ordered such that:

$$\operatorname{Re}(\rho\omega_1) \leq \operatorname{Re}(\rho\omega_2), \quad \forall \rho \in \mathcal{S}_k. \quad (3.8)$$

Specifically, in the sector $\mathcal{S}_0$, we choose:

$$\omega_1 := e^{i\frac{\pi}{2}} = i, \quad \omega_2 := e^{i\frac{3\pi}{2}} = -i,$$

and reorder them in each of the remaining sectors to ensure that inequality (3.8) holds.

Setting $\mu := \rho\omega_1$ for ρ in each sector $\mathcal{S}_k$, we recall the following result concerning the fundamental solutions of system (3.5). For more details, see [1, 6].

Lemma 3.2. In each sector $\mathcal{S}_k$, for $\rho \in \mathcal{S}_0$ with $|\rho|$ large enough, there are four linearly independent solutions $y_s(z)$ for $s = 1, 2, 3, 4$ of the equation:

$$f^{(4)}(z) + a_1(z)f'''(z) + a_2(z)f''(z) + a_3(z)f'(z) \\ + \rho^2[f''(z) + b_1(z)f'(z) - b_2(z)f(z)] = 0,$$

such that for $j = 0, 1, 2, 3$,

$$y_s^{(j)}(z; \rho) = h_s^{(j)}(z) + \mathcal{O}(\rho^{-1}), \quad s = 1, 2, \\ y_s^{(j)}(z; \rho) = (\rho\omega_{s-2})^j e^{\rho\omega_{s-2}z} (y_0(z) + \mathcal{O}(\rho^{-1})), \quad (3.9) \\ s = 3, 4,$$

where

$$y_0(z) := \exp\left(\int_0^z \frac{1}{2}[b_1(t) - a_1(t)]dt\right), \\ y_0(0) = 1,$$

and the following equation:

$$f''(z) + b_1(z)f'(z) - b_2(z)f(z) = 0,$$

has two linearly independent solutions:

$$h_1(z) := h_1(x(z)), \quad h_2(z) := h_2(x(z)).$$

For convenience, we denote $[a]_1 = a + \mathcal{O}(\rho^{-1})$ and use the fundamental solutions (3.9) in the boundary conditions B_1, B_2, B_3 and B_4 . As $|\rho|$ becomes sufficiently large, the eigenvalue problem (3.5) satisfies the following asymptotic expressions:

$$B_4(y_s, \rho) = \begin{cases} h_s(0) + \mathcal{O}(\rho^{-1}) := [h_s(0)]_1, & s = 1, 2, \\ 1 + \mathcal{O}(\rho^{-1}) := [1]_1, & s = 3, 4, \end{cases}$$

$$B_3(y_s, \rho) = \begin{cases} [x_z(0)h_s'(0)]_1, & s = 1, 2, \\ \rho\omega_{s-2}[1]_1, & s = 3, 4, \end{cases}$$

$$B_2(y_s, \rho) = y_s''(1, \rho) + \frac{c_{22}}{c_{21}}y_s'(1, \rho) + i\frac{c_{23}}{c_{21}}\kappa_{12}\rho h^{-1}y_s'(1, \rho) \\ + \frac{c_{23}}{c_{21}}\alpha y_s'(1, \rho) + i\frac{1}{c_{21}}\kappa_{11}\rho h^{-1}y_s(1, \rho), \\ = \begin{cases} \rho \left[i\frac{c_{23}}{c_{21}}\kappa_{12}h^{-1}x_z(1)h_s'(1) + i\frac{1}{c_{21}}\kappa_{11}h^{-1}h_s(1) \right]_1, & s = 1, 2, \\ \rho^2 e^{\rho\omega_{s-2}} \left[\left(\omega_{s-2} + i\frac{c_{23}}{c_{21}}\kappa_{12}h^{-1} \right) y_0(1)\omega_{s-2} \right]_1, & s = 3, 4, \end{cases}$$

$$B_1(y_s, \rho) = y_s'''(1, \rho) + \frac{c_{12}}{c_{11}}y_s''(1, \rho) + \frac{c_{13}}{c_{11}}y_s'(1, \rho) \\ + \frac{c_{14}}{c_{11}}\rho^2 h^{-2}y_s'(1, \rho) - i\kappa_{22}\frac{c_{15}}{c_{11}}\rho h^{-1}y_s'(1, \rho) \\ - i\frac{1}{c_{11}}\kappa_{21}\rho h^{-1}y_s(1, \rho) - \frac{\beta}{c_{11}}y_s(1, \rho), \\ = \begin{cases} \rho^2 [x_z(1)h_s'(1)]_1, & s = 1, 2, \\ \rho^3 e^{\rho\omega_{s-2}} [y_0(1)\omega_{s-2}^3 + y_0(1)\omega_{s-2}]_1, & s = 3, 4. \end{cases}$$

The characteristic determinant of (3.5) is given by:

$$\Delta(\rho) = \begin{vmatrix} B_4(y_1, \rho) & B_4(y_2, \rho) & B_4(y_3, \rho) & B_4(y_4, \rho) \\ B_3(y_1, \rho) & B_3(y_2, \rho) & B_3(y_3, \rho) & B_3(y_4, \rho) \\ B_2(y_1, \rho) & B_2(y_2, \rho) & B_2(y_3, \rho) & B_2(y_4, \rho) \\ B_1(y_1, \rho) & B_1(y_2, \rho) & B_1(y_3, \rho) & B_1(y_4, \rho) \end{vmatrix}. \quad (3.10)$$

Next, we expand the characteristic determinant (3.10) in the sector $\mathcal{S}_0$. By substituting $(B_i(y_s, \rho))_{i=4,3,2,1,s=1,2,3,4}$ into (3.10) such that $\omega_1 = i$ and $\omega_2 = -i$, we find

$$\Delta(\rho) = \begin{vmatrix} [h_1(0)]_1 & [h_2(0)]_1 \\ [x_z(0)h'_1(0)]_1 & [x_z(0)h'_2(0)]_1 \\ i\rho \left[\frac{c_{23}}{c_{21}} \kappa_{12} h^{-1} x_z(1)h'_1(1) + \frac{\kappa_{11}}{c_{21}} h^{-1} h_1(1) \right]_1 & i\rho \left[\frac{c_{23}}{c_{21}} \kappa_{12} h^{-1} x_z(1)h'_2(1) + \frac{\kappa_{11}}{c_{21}} h^{-1} h_2(1) \right]_1 \\ \rho^2 [x_z(1)h'_1(1)]_1 & \rho^2 [x_z(1)h'_2(1)]_1 \\ [1]_1 & [1]_1 \\ i\rho[1]_1 & -i\rho[1]_1 \\ \rho^2 e^{\rho\omega_1} \left[\left(-1 - \frac{c_{23}}{c_{21}} \kappa_{12} h^{-1} \right) y_0(1) \right]_1 & \rho^2 e^{\rho\omega_2} \left[\left(-1 + \frac{c_{23}}{c_{21}} \kappa_{12} h^{-1} \right) y_0(1) \right]_1 \\ \rho^3 e^{\rho\omega_1} [0]_1 & \rho^3 e^{\rho\omega_2} [0]_1 \end{vmatrix}.$$

Expanding the determinant $\Delta(\rho)$ along the third column and the second row, we obtain:

$$\Delta(\rho) = -i\rho^5 e^{\rho\omega_2} x_z(1) \left([h_1(0)]_1 \times [h'_2(1)]_1 - [h_2(0)]_1 \times [h'_1(1)]_1 \right) \times \left[\left(1 - \frac{c_{23}}{c_{21}} \kappa_{12} h^{-1} \right) y_0(1) \right]_1 \\ - i\rho^5 e^{\rho\omega_1} x_z(1) \left([h_1(0)]_1 \times [h'_2(1)]_1 - [h_2(0)]_1 \times [h'_1(1)]_1 \right) \times \left[\left(1 + \frac{c_{23}}{c_{21}} \kappa_{12} h^{-1} \right) y_0(1) \right]_1.$$

Factorizing yields the following result:

$$\Delta(\rho) = -i\rho^5 y_0(1) x_z(1) \left([h_1(0)]_1 \times [h'_2(1)]_1 - [h_2(0)]_1 \times [h'_1(1)]_1 \right) \times \left(\left[1 + \frac{c_{23}}{c_{21}} \kappa_{12} h^{-1} \right]_1 e^{\rho\omega_1} + \left[1 - \frac{c_{23}}{c_{21}} \kappa_{12} h^{-1} \right]_1 e^{\rho\omega_2} \right). \quad (3.11)$$

Clearly, we have:

$$\frac{c_{14}}{c_{11}} = \frac{I_r(1)z_x(1)}{z_x^3 EI(1)} = \frac{I_r(1)}{z_x^2 EI(1)} = h^2.$$

From the asymptotic expansion of the characteristic determinant (3.11), we recall the notions of regularity, strong regularity and the normalized boundary conditions of problem (3.5) (see [9, 10]).

Definition 3.1. [10] For $\rho \in \mathcal{S}_0$, an asymptotic expansion of the characteristic determinant $\Delta(\rho)$ is given by:

$$\text{if } n - p = 2k - 1, \\ \Delta(\rho) = \rho^{X_p} e^{\rho\Omega} \left([\Theta_0(\rho)]_l + [\Theta_1(\rho)]_l e^{\rho\omega_k} \right),$$

and since $\omega_{k+1} = -\omega_k$,

$$\text{if } n - p = 2k, \\ \Delta(\rho) = \rho^{X_p} e^{\rho\Omega} \left([\Theta_0(\rho)]_l + [\Theta_1(\rho)]_l e^{\rho\omega_k} \right. \\ \left. + [\Theta_2(\rho)]_l e^{2\rho\omega_k} \right),$$

with

$$X_p := l_{p+1} + \dots + l_n, \\ \Omega := \omega_{k+1} + \dots + \omega_{n-p}, \\ \Theta_i(\rho) = \Theta_{i0} + \frac{\Theta_{i1}}{\rho} + \dots + \frac{\Theta_{il}}{\rho^l}, \quad i = 0, 1, 2.$$

Definition 3.2. [10] A regular problem is said to be strongly regular if either $n - p = 2k - 1$ or $n - p = 2k$

and the quadratic equation:

$$\Theta_{00} + \Theta_{10}\zeta + \Theta_{20}\zeta^2 = 0, \quad (3.12)$$

has two different roots $\zeta_1, \zeta_2 \in \mathbb{C}$.

The following theorem formalizes the results discussed above.

Theorem 3.1. In the sector $\mathcal{S}_0$, the characteristic determinant $\Delta(\rho)$ of the eigenvalue problem (3.5) has the following asymptotic expansion:

$$\Delta(\rho) = -i\rho^5 y_0(1) x_z(1) \Gamma \left([1 + \kappa_{12}\chi]_1 e^{i\rho} \right. \\ \left. + [1 - \kappa_{12}\chi]_1 e^{-i\rho} \right), \quad (3.13)$$

where

$$\Gamma = h'_2(0)h_1(1) - h'_1(1)h_2(0)$$

is the nonzero determinant defined in (3.7) and

$$\chi = (I_r(1)EI(1))^{-1/2}.$$

Furthermore, the eigenvalue problem (3.5) is strongly regular if and only if the following condition holds:

$$1 - \kappa_{12}\chi \neq 0. \quad (3.14)$$

Proof. First, from the characteristic determinant (3.11), we compute the following:

$$\frac{c_{23}}{c_{21}} = \frac{z_x(1)}{z_x^2(1)EI(1)} = h(I_r(1)EI(1))^{-1/2} = h\chi.$$

By taking $\omega_1 = i$ and $\omega_2 = -i$ in the sector $\mathcal{S}_0$ and substituting into (3.11), we obtain the expression (3.13). Secondly, according to Definition 3.1, we have:

$$\Theta_{00} = 0, \quad \Theta_{10} = 1 + \kappa_{12}\chi,$$

and $\Theta_{20} = 1 - \kappa_{12}\chi$.

Since

$$x_z(1) > 0, \quad y_0(1) > 0, \quad \text{and} \quad \Gamma \neq 0,$$

equation (3.12) has two distinct roots if and only if

$$1 - \kappa_{12}\chi \neq 0.$$

Consequently, strong regularity can be verified directly. Therefore, the zeros of the characteristic determinant $\Delta(\rho)$ are simple when their moduli are sufficiently large.

It is important to observe that the eigenvalues found in sectors $\mathcal{S}_1$ and $\mathcal{S}_3$ coincide with those of $\mathcal{S}_0$ and $\mathcal{S}_2$ (see [1, 14]). The following theorem concludes this section by summarizing the spectral properties of problem (3.5).

Theorem 3.2.

Let $\mathcal{A}$ be the operator defined by (2.4)–(2.5). Assume that condition (3.14) is fulfilled. Then there exists an integer $N > 0$ such that the eigenvalues λ_k of the problem (3.5) have the following asymptotic expression:

$$\lambda_k = \frac{1}{h} \left(\frac{1}{2} \ell_0 \pm k\pi i \right) + \mathcal{O}(k^{-1}), \quad (3.15)$$

$k \geq N, \quad k \in \mathbb{N},$

with

$$h = \int_0^1 \left(\frac{I_r(\xi)}{EI(\xi)} \right)^{\frac{1}{2}} d\xi,$$

and

$$\ell_0 = \begin{cases} \ln \left(\frac{\kappa_{12}\chi - 1}{\kappa_{12}\chi + 1} \right), & \kappa_{12}\chi > 1 \\ \ln \left(\frac{1 - \kappa_{12}\chi}{\kappa_{12}\chi + 1} \right) + \pi i, & \kappa_{12}\chi < 1. \end{cases} \quad (3.16)$$

Moreover, we have:

$$\operatorname{Re}(\ell_0) = \ln \left| \frac{\kappa_{12}\chi - 1}{\kappa_{12}\chi + 1} \right| < 0,$$

and

$$\lim_{k \rightarrow +\infty} \operatorname{Re}(\lambda_k) = \frac{1}{2h} \operatorname{Re}(\ell_0). \quad (3.17)$$

Proof. We begin by considering the sector $\mathcal{S}_0$. Assuming $\Delta(\rho) = 0$, it follows from (3.13) that:

$$(1 + \kappa_{12}\chi)e^{i\rho} + (1 - \kappa_{12}\chi)e^{-i\rho} + \mathcal{O}(\rho^{-1}) = 0. \quad (3.18)$$

The corresponding lower-order equation is given by:

$$(1 + \kappa_{12}\chi)e^{i\rho} + (1 - \kappa_{12}\chi)e^{-i\rho} = 0. \quad (3.19)$$

Multiplying this equation by $e^{i\rho}$ and rearranging terms, we obtain:

$$e^{2i\rho} = \frac{\kappa_{12}\chi - 1}{\kappa_{12}\chi + 1}. \quad (3.20)$$

Case 1: If $\kappa_{12}\chi > 1$, then (3.20) can be written as:

$$e^{2i\rho} = \frac{\kappa_{12}\chi - 1}{\kappa_{12}\chi + 1} e^0.$$

By taking the complex logarithm, we find:

$$2i\rho_k = \ln \left(\frac{\kappa_{12}\chi - 1}{\kappa_{12}\chi + 1} \right) + 2k\pi i, \quad k = 1, 2, \dots$$

Thus, the roots ρ_k satisfy:

$$i\rho_k = \frac{1}{2}\ell_0 + k\pi i, \quad k = 1, 2, \dots$$

Case 2: If $\kappa_{12}\chi < 1$, equation (3.20) becomes:

$$e^{2i\rho} = \frac{1 - \kappa_{12}\chi}{\kappa_{12}\chi + 1} e^{\pi i}.$$

Applying the complex logarithm yields:

$$2i\rho_k = \ln \left(\frac{1 - \kappa_{12}\chi}{\kappa_{12}\chi + 1} \right) + \pi i + 2k\pi i, \quad k = 1, 2, \dots,$$

which implies that the roots ρ_k satisfy:

$$2i\rho_k = \ell_0 + 2k\pi i, \quad k = 1, 2, \dots,$$

where ℓ_0 is defined in (3.16). Consequently, the solutions $\widetilde{\mu}_k = i\rho_k$ of (3.19) in the sector $\mathcal{S}_0$ are given by:

$$\widetilde{\mu}_k = i\rho_k = \frac{1}{2}\ell_0 + k\pi i, \quad k = 1, 2, \dots$$

By applying Rouché's theorem, the solutions μ_k of the full equation (3.18) satisfy, for a sufficiently large positive integer N :

$$\mu_k = \frac{1}{2}\ell_0 + k\pi i + \mathcal{O}(k^{-1}), \quad (3.21)$$

$k \geq N, k \in \mathbb{N}.$

In sector $\mathcal{S}_2$, we take $\omega_1 := -i$ and $\omega_2 := i$ such that the inequality (3.8) is verified. Then, we substitute them in the boundary conditions B_1, B_2, B_3 and B_4 to obtain the characteristic determinant in sector $\mathcal{S}_2$. Expanding this determinant, we have:

$$\Delta(\rho) = i\rho^5 y_0(1) x_z(1) \Gamma \left([1 - \kappa_{12}\chi]_1 e^{i\rho} + [1 + \kappa_{12}\chi]_1 e^{-i\rho} \right).$$

The characteristic equation $\Delta(\rho) = 0$ then becomes:

$$(1 - \kappa_{12}\chi)e^{i\rho} + (1 + \kappa_{12}\chi)e^{-i\rho} + \mathcal{O}(\rho^{-1}) = 0.$$

Following the same approach as before, the leading-order equation is:

$$(1 - \kappa_{12}\chi) e^{i\rho} + (1 + \kappa_{12}\chi) e^{-i\rho} = 0.$$

Multiplying this equation by $e^{i\rho}$ and rearranging, we obtain:

$$e^{2i\rho} = \frac{\kappa_{12}\chi + 1}{\kappa_{12}\chi - 1}.$$

This yields the roots:

$$i\rho_k = -\frac{1}{2}\ell_0 + k\pi i, \quad k = 1, 2, \dots.$$

Recall that in sector $\mathcal{S}_2$, the relationship is $\tilde{\mu} = \rho\omega_1 = -i\rho$. Consequently,

$$\tilde{\mu}_k = \frac{1}{2}\ell_0 - k\pi i, \quad k = 1, 2, \dots.$$

For a sufficiently large positive integer N , we have:

$$\begin{aligned} \mu_k &= \frac{1}{2}\ell_0 - k\pi i + \mathcal{O}(k^{-1}), \\ k &\geq N, \quad k \in \mathbb{N}. \end{aligned} \quad (3.22)$$

Finally, using the relation $\mu = h\lambda$, expressions (3.21) and (3.22) yield the desired asymptotic distribution of the eigenvalues:

$$\lambda_k = \frac{1}{h} \left(\frac{1}{2}\ell_0 + k\pi i \right) + \mathcal{O}(k^{-1}), \quad k = \pm 1, \pm 2, \dots.$$

4. Riesz basis property and stability results

This section is devoted to proving the Riesz basis property of the eigenfunctions of $\mathcal{A}$ and investigating the stability of system (2.6). As the asymptotic behavior of the eigenvalues λ_k in (3.15) is consistent with that in [6], we adopt the methodology presented therein. We begin by establishing the following estimate for the resolvent operator.

Lemma 4.1. *Let $\mathcal{A}$ be the operator defined by (2.4)–(2.5), and let $\{\nu_k\}_{k \in \mathbb{Z}}$ denote its eigenvalues. For any $\delta > 0$, there exists a constant $M > 0$ such that the resolvent operator $R(\lambda, \mathcal{A})$ satisfies:*

$$\|R(\lambda, \mathcal{A})\|_{\mathbb{H}} \leq M|\lambda|^2$$

for all $\lambda \in \rho(\mathcal{A})$ such that $|\lambda - \nu_k| > \delta$.

Proof. See [6] for details.

We have the following theorem on the completeness of the generalized eigenfunctions of the operator $\mathcal{A}$.

Theorem 4.1. *Let $\mathcal{A}$ be defined as in (2.4)–(2.5). If $1 - \kappa_{12}\chi \neq 0$, then the system of generalized eigenfunctions of $\mathcal{A}$ is complete in $\mathbb{H}$.*

Proof. See [6] for details.

Theorems 3.2 and 4.1 ensure that all conditions of Theorem 4.2 in [6] are satisfied. Consequently, we obtain the following result regarding the growth condition.

Theorem 4.2. *Assume that $1 - \kappa_{12}\chi \neq 0$. Then, the system (2.6) is a Riesz-spectral system (i.e., the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis in $\mathbb{H}$). Furthermore, it satisfies the spectrum-determined growth condition.*

The above theorem guarantees that the optimal decay rate of the energy, denoted by $\omega(\mathcal{A})$, is determined by the spectral abscissa $s(\mathcal{A})$. More precisely,

$$\omega(\mathcal{A}) = s(\mathcal{A}) = \sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(\mathcal{A})\}. \quad (4.1)$$

We are now in a position to investigate the exponential stability of system (2.6). Since (3.15) describes the asymptote of $\sigma(\mathcal{A})$, for any small $\epsilon > 0$, there are only finitely many eigenvalues of $\mathcal{A}$ in the following half-plane (see [15]):

$$\operatorname{Re}(\lambda) > \frac{1}{2h}\operatorname{Re}(\ell_0) + \epsilon.$$

Motivated by the result (4.1), the system (1.1) is exponentially stable if and only if $s(\mathcal{A}) < 0$. The following lemma demonstrates that $\operatorname{Re}(\lambda) < 0$ under specific conditions.

Lemma 4.2. *Let $\lambda \in \sigma(\mathcal{A})$ be the eigenvalue of $\mathcal{A}$ satisfying (3.1). If $\kappa_{22} = \kappa_{11}$, then, there exists $\omega_0 > 0$ such that:*

$$\operatorname{Re}(\lambda) = -\omega_0 < 0,$$

where

$$\begin{aligned} \omega_0 &= \frac{\kappa_{21}|\phi(1)|^2 + \kappa_{12}|\phi'(1)|^2}{2 \int_0^1 [r(x)|\phi(x)|^2 + I_r(x)|\phi'(x)|^2] dx} \\ &\quad + \frac{(\kappa_{11} + \kappa_{22})\operatorname{Re}(\phi'(1)\overline{\phi(1)})}{2 \int_0^1 [r(x)|\phi(x)|^2 + I_r(x)|\phi'(x)|^2] dx}. \end{aligned} \quad (4.2)$$

Proof. Let (λ, ϕ) satisfy the characteristic equation (3.1):

$$\lambda^2 r(x)\phi(x) - \lambda^2 (I_r(x)\phi'(x))' + (EI(x)\phi''(x))'' = 0. \quad (4.3)$$

Multiplying (4.3) by $\overline{\phi}$ and integrating over $(0, 1)$ with respect to x , we get:

$$\begin{aligned} \lambda^2 \int_0^1 r(x)|\phi(x)|^2 dx - \lambda^2 \int_0^1 (I_r(x)\phi'(x))' \overline{\phi(x)} dx \\ + \int_0^1 (EI(x)\phi''(x))'' \overline{\phi(x)} dx = 0. \end{aligned}$$

Then, performing a double integration by parts, we have:

$$\begin{aligned} & \lambda^2 \int_0^1 r(x)|\phi(x)|^2 dx + \lambda^2 \int_0^1 I_r(x)|\phi'(x)|^2 dx \\ & + \int_0^1 EI(x)|\phi''(x)|^2 dx + \kappa_{21}\lambda|\phi(1)|^2 \\ & + \kappa_{22}\lambda\phi'(1)\overline{\phi(1)} + \beta|\phi(1)|^2 \\ & + \kappa_{11}\lambda\phi(1)\overline{\phi'(1)} + \kappa_{12}\lambda|\phi'(1)|^2 \\ & + \alpha|\phi'(1)|^2 = 0. \end{aligned} \quad (4.4)$$

Consider the following algebraic forms:

$$\begin{aligned} \lambda &= \mathcal{R}e\lambda + i\mathcal{I}m\lambda, \\ \phi'(1)\overline{\phi(1)} &= \mathcal{R}e(\phi'(1)\overline{\phi(1)}) + i\mathcal{I}m(\phi'(1)\overline{\phi(1)}). \end{aligned}$$

After some computations on (4.4), we separate the real and imaginary parts. Indeed,

$$\begin{aligned} & ((\mathcal{R}e\lambda)^2 - (\mathcal{I}m\lambda)^2) \int_0^1 [r(x)|\phi(x)|^2 + I_r(x)|\phi'(x)|^2] dx \\ & + \kappa_{21}(\mathcal{R}e\lambda)|\phi(1)|^2 + \kappa_{12}(\mathcal{R}e\lambda)|\phi'(1)|^2 \\ & + \kappa_{22}(\mathcal{R}e\lambda\mathcal{R}e(\phi'(1)\overline{\phi(1)}) - \mathcal{I}m\lambda\mathcal{I}m(\phi'(1)\overline{\phi(1)})) \\ & + \kappa_{11}(\mathcal{R}e\lambda\mathcal{R}e(\phi(1)\overline{\phi'(1)}) - \mathcal{I}m\lambda\mathcal{I}m(\phi(1)\overline{\phi'(1)})) \\ & + \int_0^1 EI(x)|\phi''(x)|^2 dx + \beta|\phi(1)|^2 + \alpha|\phi'(1)|^2 = 0, \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} & 2(\mathcal{R}e\lambda)(\mathcal{I}m\lambda) \int_0^1 [r(x)|\phi(x)|^2 + I_r(x)|\phi'(x)|^2] dx \\ & + \kappa_{22}(\mathcal{R}e\lambda\mathcal{I}m(\phi'(1)\overline{\phi(1)}) + \mathcal{I}m\lambda\mathcal{R}e(\phi'(1)\overline{\phi(1)})) \\ & + \kappa_{11}(\mathcal{R}e\lambda\mathcal{I}m(\phi(1)\overline{\phi'(1)}) + \mathcal{I}m\lambda\mathcal{R}e(\phi(1)\overline{\phi'(1)})) \\ & + \kappa_{21}(\mathcal{I}m\lambda)|\phi(1)|^2 + \kappa_{12}(\mathcal{I}m\lambda)|\phi'(1)|^2 = 0. \end{aligned} \quad (4.6)$$

If $\mathcal{I}m\lambda = 0$, then by (4.5) and Lemma 2.1, we have:

$$\begin{aligned} & (\mathcal{R}e\lambda)^2 \int_0^1 [r(x)|\phi(x)|^2 + I_r(x)|\phi'(x)|^2] dx \\ & + (\mathcal{R}e\lambda)(\kappa_{21}|\phi(1)|^2 + \kappa_{12}|\phi'(1)|^2) \\ & + (\mathcal{R}e\lambda)(\kappa_{11} + \kappa_{22})\mathcal{R}e(\phi'(1)\overline{\phi(1)}) \\ & + \int_0^1 EI(x)|\phi''(x)|^2 dx + \alpha|\phi'(1)|^2 + \beta|\phi(1)|^2 = 0. \end{aligned}$$

Thus $\mathcal{R}e(\lambda) < 0$, because from (2.7) of Lemma 2.2, we know that:

$$\kappa_{21}|\phi(1)|^2 + (\kappa_{11} + \kappa_{22})\mathcal{R}e(\phi'(1)\overline{\phi(1)}) + \kappa_{12}|\phi'(1)|^2 > 0.$$

Now, suppose that $\mathcal{I}m(\lambda) \neq 0$.

According to Lemma 2.1, we have:

$$\begin{aligned} \mathcal{R}e(\phi(1)\overline{\phi'(1)}) &= \mathcal{R}e(\phi'(1)\overline{\phi(1)}) \\ \mathcal{I}m(\phi(1)\overline{\phi'(1)}) &= -\mathcal{I}m(\phi'(1)\overline{\phi(1)}). \end{aligned}$$

Inserting these identities into (4.6) gives:

$$\begin{aligned} & 2\mathcal{R}e\lambda\mathcal{I}m\lambda \int_0^1 [r(x)|\phi(x)|^2 + I_r(x)|\phi'(x)|^2] dx \\ & + (\kappa_{22} - \kappa_{11})\mathcal{R}e\lambda\mathcal{I}m(\phi'(1)\overline{\phi(1)}) \\ & + (\kappa_{11} + \kappa_{22})\mathcal{I}m\lambda\mathcal{R}e(\phi'(1)\overline{\phi(1)}) \\ & + \kappa_{21}(\mathcal{I}m\lambda)|\phi(1)|^2 + \kappa_{12}(\mathcal{I}m\lambda)|\phi'(1)|^2 = 0. \end{aligned} \quad (4.7)$$

Consequently, if $\mathcal{I}m\lambda \neq 0$ and $\kappa_{22} = \kappa_{11}$, then equation (4.7) becomes:

$$\begin{aligned} & (\mathcal{I}m\lambda) [\kappa_{21}|\phi(1)|^2 + \kappa_{12}|\phi'(1)|^2 \\ & + 2(\mathcal{R}e\lambda) \int_0^1 [r(x)|\phi(x)|^2 + I_r(x)|\phi'(x)|^2] dx \\ & + (\kappa_{11} + \kappa_{22})\mathcal{R}e(\phi'(1)\overline{\phi(1)})] = 0. \end{aligned}$$

We deduce that:

$$\begin{aligned} \mathcal{R}e\lambda &= - \frac{\kappa_{21}|\phi(1)|^2 + \kappa_{12}|\phi'(1)|^2}{2 \int_0^1 [r(x)|\phi(x)|^2 + I_r(x)|\phi'(x)|^2] dx} \\ & - \frac{(\kappa_{11} + \kappa_{22})\mathcal{R}e(\phi'(1)\overline{\phi(1)})}{2 \int_0^1 [r(x)|\phi(x)|^2 + I_r(x)|\phi'(x)|^2] dx}. \end{aligned}$$

So, $\mathcal{R}e\lambda < 0$ because

$$\begin{aligned} & \kappa_{21}|\phi(1)|^2 + (\kappa_{11} + \kappa_{22})\mathcal{R}e(\phi'(1)\overline{\phi(1)}) \\ & + \kappa_{12}|\phi'(1)|^2 > 0. \end{aligned}$$

Summarizing the discussion above, we give the main theorem of this paper.

Theorem 4.3. Assume $1 - \kappa_{12}\chi \neq 0$ and $\kappa_{22} = \kappa_{11}$. Then, under the conditions (1.4), the system (2.6) is exponentially stable. Therefore, there are two strictly positive constants M and γ such that:

$$E(t) \leq ME(0)e^{-\gamma t}, \quad \forall t \geq 0.$$

Proof. Theorem 4.2 ensures that:

$\omega(\mathcal{A}) = s(\mathcal{A}) = \sup\{\mathcal{R}e(\lambda) : \lambda \in \sigma(\mathcal{A})\}$. Lemma 4.2 establishes that $\mathcal{R}e(\lambda) < 0$, while Theorem 3.2 shows that the imaginary axis is not an asymptote for the spectrum. Consequently, we have $s(\mathcal{A}) < 0$, which completes the proof.

Remark 4.1. The exponential stability demonstrated in Theorem 4.3 is of significant interest in engineering and physical sciences, as it guarantees a minimum decay rate of perturbations toward the equilibrium state. This work generalizes previous studies by incorporating more comprehensive boundary conditions (1.3). Indeed, the results in [1, 4–8] can be recovered as special cases by setting certain controls to zero.

5. Numerical study

This section is devoted to the numerical approximation of problem (1.1) using the finite element method (cf. [16–20]). After defining a finite-dimensional subspace of H_e^2 , we seek the corresponding Galerkin approximation. Finally, we present numerical experiments to illustrate our results.

5.1. Semi-discrete scheme

The first step involves deriving the weak formulation of the boundary value problem (1.1). By multiplying equation (1.1) by a test function $\phi(x) \in H_e^2$ and following the approach described in (2.1), we obtain the following weak form:

$$\begin{aligned} & \int_0^1 r(x)y_{tt}(x)\phi(x)dx + \int_0^1 I_r(x)y_{ttx}(x)\phi_x(x)dx \\ & + \int_0^1 EI(x)y_{xx}(x)\phi_{xx}(x)dx \\ & + (\kappa_{11}y_t(1) + \kappa_{12}y_{xt}(1) + \alpha y_x(1))\phi_x(1) \\ & + (\kappa_{21}y_t(1) + \kappa_{22}y_{xt}(1) + \beta y(1))\phi(1) = 0. \end{aligned} \quad (5.1)$$

The second step is the choice of a finite-dimensional subspace. Let us subdivide the interval $[0, 1]$ into n intervals $[x_i, x_{i+1}]$ of the same length $h = x_{i+1} - x_i$ as in [21]. Let $\mathcal{V}_h \subset H_e^2$ be the finite-dimensional space spanned by the Hermite polynomials $\{\phi_i\}_{i=1}^N$ (with $N = 2n$), which form a basis of $\mathcal{V}_h$. The Galerkin finite element approximation of problem (5.1) is defined as follows:

Definition 5.1. *The finite element approximation of (5.1) consists in finding $y_h \in C^2([0, \infty); \mathcal{V}_h)$ such that:*

$$\begin{aligned} & \int_0^1 r(x)(y_h)_{tt}\phi_i dx + \int_0^1 I_r(x)(y_h)_{ttx}(\phi_i)_x dx \\ & + \int_0^1 EI(x)(y_h)_{xx}(\phi_i)_{xx} dx + \beta y_h(1)\phi_i(1) \\ & + \kappa_{12}(y_h)_{xt}(1)(\phi_i)_x(1) + \kappa_{11}(y_h)_t(1)(\phi_i)_x(1) \\ & + \alpha(y_h)_x(1)(\phi_i)_x(1) + \kappa_{21}(y_h)_t(1)\phi_i(1) \\ & + \kappa_{22}(y_h)_{xt}(1)\phi_i(1) = 0. \end{aligned} \quad (5.2)$$

For all $\phi_i \in \mathcal{V}_h$, with the initial conditions:

$$y_h(\cdot, 0) = y_{h,0} \in \mathcal{V}_h, (y_h)_t(\cdot, 0) = y_{h,1} \in \mathcal{V}_h.$$

Equation (5.2) can be transformed into a system of second-order ordinary differential equations in time by expanding the solution in the basis of $\mathcal{V}_h$. Specifically, we look for a function $y_h \in \mathcal{V}_h$ of the form:

$$y_h(x, t) = \sum_{j=1}^N Y_j(t)\phi_j(x), \quad (5.3)$$

where $Y(t) = (Y_j(t))_{1 \leq j \leq N}$ is the vector of time-dependent coefficients. Substituting this expansion into (5.2), the problem can be expressed in matrix form as:

$$\mathbf{M}Y_{tt} + \mathbf{S}Y_t + \mathbf{K}Y = 0, \quad (5.4)$$

where

$$\mathbf{M} = (m_{ij})_{1 \leq i, j \leq N},$$

$$\mathbf{S} = (s_{ij})_{1 \leq i, j \leq N},$$

$$\mathbf{K} = (k_{ij})_{1 \leq i, j \leq N},$$

are given by:

$$\begin{aligned} m_{ij} &= \int_0^1 (r(x)\phi_i\phi_j + I_r(x)(\phi_i)_x(\phi_j)_x)dx, \\ s_{ij} &= (\kappa_{11}\phi_j(1) + \kappa_{12}(\phi_j)_x(1))(\phi_i)_x(1) \\ &\quad + (\kappa_{21}\phi_j(1) + \kappa_{22}(\phi_j)_x(1))\phi_i(1), \\ k_{ij} &= \int_0^1 EI(x)(\phi_i)_{xx}(\phi_j)_{xx}dx \\ &\quad + \beta\phi_i(1)\phi_j(1) + \alpha(\phi_i)_x(\phi_j)_x. \end{aligned}$$

Since the matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite, they are invertible. It follows that problem (5.2) has a unique solution $y_h \in C^2([0, \infty); \mathcal{V}_h)$. Now, we rewrite equation (5.4) as a system of first-order ordinary differential equations in time:

$$\frac{d}{dt}\mathbb{Y} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{S} \end{pmatrix} \mathbb{Y},$$

where $\mathbb{Y} = \begin{pmatrix} Y \\ Y_t \end{pmatrix}$. Then, the matrix:

$$\mathbb{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{S} \end{pmatrix}$$

is the spectral matrix. $\mathbf{0}$ and $\mathbf{I}$ denote the zero and identity matrices, respectively.

5.2. Numerical experiments

The numerical studies below allow us to graphically analyze the asymptotic behavior of the system (Figures 1 to 8). For $r = EI = Ir = 1$ and under the condition (1.4), the spectral matrix $\mathbb{A}$ is implemented. The numerical simulations are carried out with MATLAB for $N = 100$ and $h = 1/N$.

Case 1 : Representation of the spectrum.

We fix $\alpha = \beta = 1$, $\kappa_{21} = 4$, $\kappa_{12} = 5$.

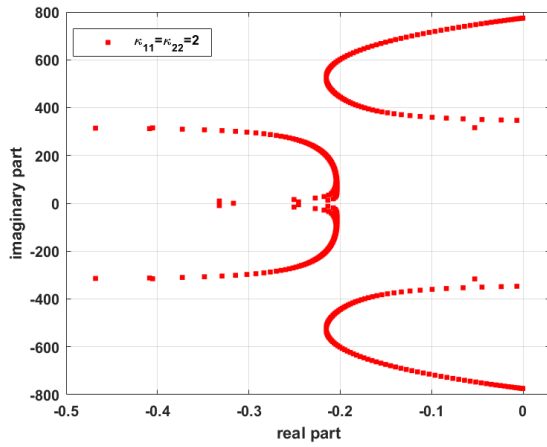


Fig. 1. Spectrum for $\kappa_{11} = \kappa_{22}$.

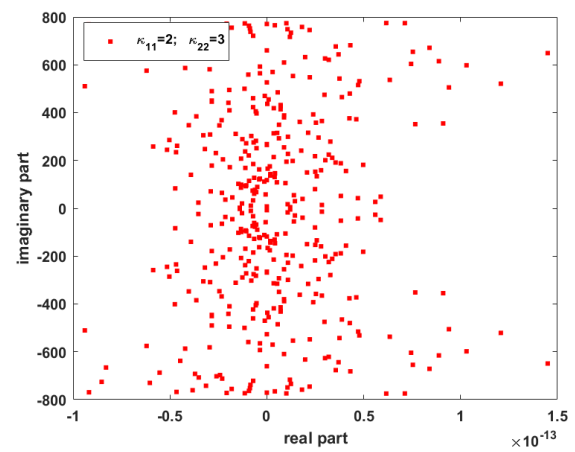


Fig. 2. Spectrum for $\kappa_{11} \neq \kappa_{22}$.

Case 2 : Influence of control parameters on the spectrum.

$\alpha = \beta = 1$, $\kappa_{11} = \kappa_{22} = 2$, $\kappa_{21} = 4$.

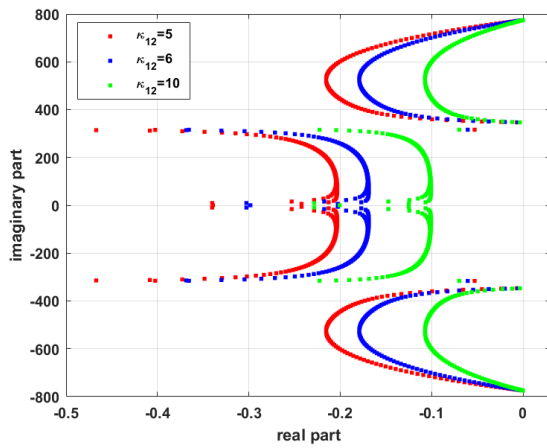


Fig. 3. Effect of κ_{12} .

$\alpha = \beta = 1$, $\kappa_{11} = \kappa_{22} = 2$, $\kappa_{12} = 5$.

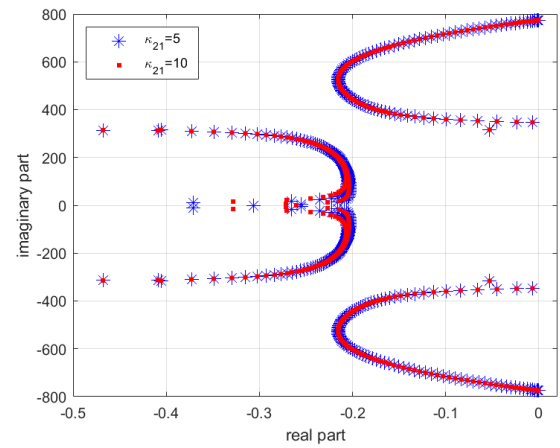


Fig. 4. Effect of κ_{21} .

$\beta = 1$, $\kappa_{11} = \kappa_{22} = 2$, $\kappa_{21} = 4$, $\kappa_{12} = 5$.

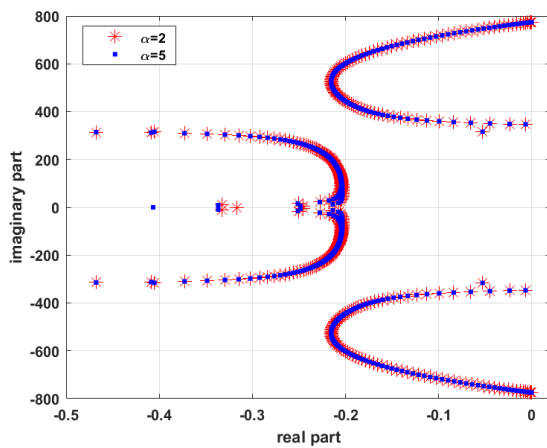


Fig. 5. Effect of α .

$\alpha = 1$, $\kappa_{11} = \kappa_{22} = 2$, $\kappa_{21} = 4$, $\kappa_{12} = 5$.

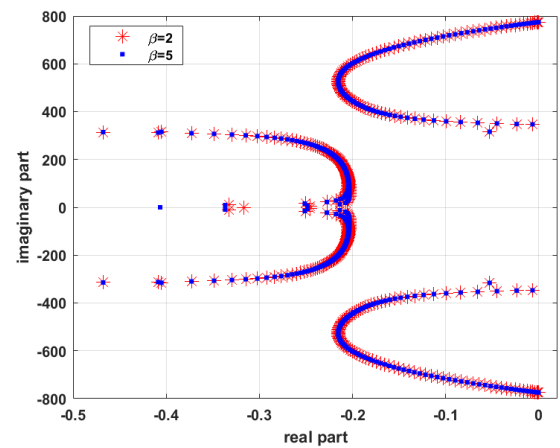
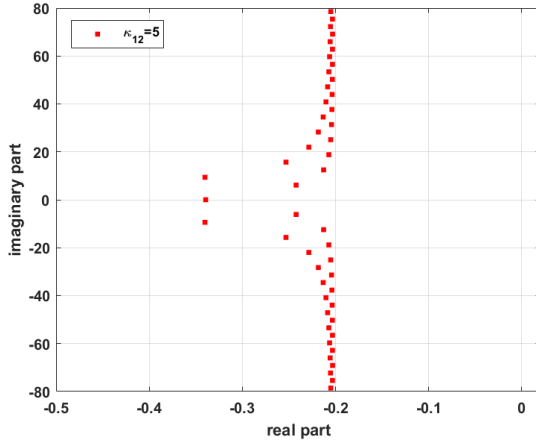


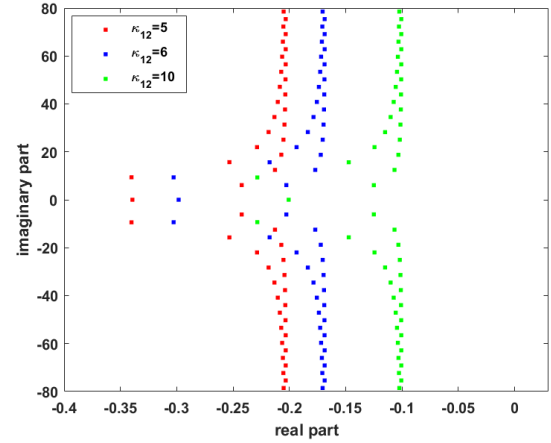
Fig. 6. Effect of β .

Case 3 : Influence of control parameters on the spectrum.

$$\alpha = \beta = 1, \kappa_{11} = \kappa_{22} = 2, \kappa_{21} = 4.$$

**Fig. 7.** Asymptote for $\kappa_{11} = \kappa_{22}$.

$$\alpha = \beta = 1, \kappa_{11} = \kappa_{22} = 2, \kappa_{21} = 4.$$

**Fig. 8.** Effect of κ_{12} on the asymptote.**Numerical results analysis:**

- The eigenvalue distribution illustrated in figure 1 reveals that all eigenvalues possess strictly negative real parts. In contrast, the spectrum in figure 2 includes positive real parts. This confirms that the system is exponentially stable under the condition $\kappa_{11} = \kappa_{22}$, in accordance with Theorem 4.3.
- Figure 3 illustrates that the spectrum shifts significantly to the right in the left complex half-plane as κ_{12} increases. Conversely, the impact of the control parameters κ_{21} , α and β on the eigenvalue distribution is negligible (Figures 4 to 6). Consequently, the moment control of the rotational velocity κ_{12} has a greater influence on the optimal rate of energy decay and appears to be the determining factor for the system's stability.
- From figure 7 we can note that the asymptote is observed at the spectral abscissa -0.2 . Under condition (1.4), the theoretical asymptote (3.17) obtained for $r = EI = Ir = 1$ is:

$$\lim_{k \rightarrow +\infty} \operatorname{Re}(\lambda_k) = \frac{1}{2h} \operatorname{Re}(\ell_0) = \frac{1}{2} \ln \left| \frac{\kappa_{12} - 1}{\kappa_{12} + 1} \right|.$$

For $\kappa_{12} = 5$, the calculated asymptote is $\lim_{k \rightarrow +\infty} \operatorname{Re}(\lambda_k) = -0.202$, which is in excellent agreement with the spectral abscissa.

6. Comparative discussion, contribution and applications of results**Previous studies :**

Earlier works [5–8] investigated the stability of non-uniform Rayleigh beams under simple boundary feedback laws. These analyses were restricted to decoupled control mechanisms, addressing only particular cases of the general approach, such as:

$$\begin{cases} L_1(y_t, y_{xt}, y_x) = (\kappa_{12}y_{xt} + \alpha y_x)(1, t), \\ L_2(y_t, y_{xt}, y) = (\kappa_{22}y_{xt} + \beta y)(1, t). \end{cases} \quad (6.1)$$

Within this framework, the authors derived the asymptotic behavior of the eigenvalues and established both the Riesz basis property and exponential stability, pro-

vided that the control gains $\kappa_{12}, \kappa_{22}, \alpha, \beta \geq 0$. Notably, these results were obtained without imposing additional structural assumptions, thereby confirming the robustness of stability under decoupled boundary feedback.

Novelty and main contributions:

The primary contribution of this study is the development of a generalized framework for coupled control laws incorporating velocity, angular velocity, and displacement, such as:

$$\begin{cases} L_1(y_t, y_{xt}, y_x) = (\kappa_{11}y_t + \kappa_{12}y_{xt} + \alpha y_x)(1, t), \\ L_2(y_t, y_{xt}, y) = (\kappa_{21}y_t + \kappa_{22}y_{xt} + \beta y)(1, t). \end{cases}$$

Through a rigorous spectral analysis, it is demonstrated that exponential stability is preserved under the critical condition $\kappa_{11} = \kappa_{22}$, despite the inherent complexity of these coupled laws. Numerical simulations based on the finite element method further validate these theoretical results, confirming that the optimal energy decay rate is determined by the spectral abscissa of the semigroup generator. This demonstrates that stability is not only preserved but also optimized under coupled boundary feedback, thereby extending the robustness of Rayleigh beam stabilization beyond the limitations of previous decoupled approaches. Given that decoupled controls are rare in practical engineering, this advancement provides a more realistic model for Rayleigh beams, ensuring that stability is maintained even under highly generalized boundary conditions.

Applications:

From an engineering standpoint, the results indicate that efficient vibration suppression of a non-uniform Rayleigh beam requires careful tuning of boundary actuators that act simultaneously on bending moments and shear forces. The Rayleigh beam equation model examined in this paper is crucial for modeling and analyzing the long-term behavior of real structures that are neither uniform nor perfectly rigid. This model finds its primary applications in fields such as aviation (aircraft wings) and robotics (flexible robot arms), where rotatory inertia effects are significant.

Comparative discussion (Table 1):**Table 1**

Comparing coupled and decoupled Rayleigh beam controls.

Model	Control Laws	Theoretical results	Numerical results	Contribution
Decoupled controls (Previous work)	Decoupled linear and angular velocities	Proven exponential stability; Riesz basis established	Energy decay may be slow or oscillatory based on parameters	Extension of the Euler-Bernoulli model with rotatory inertia
Coupled controls (Present work)	Coupled linear and angular velocities	Generalized exponential stability; key condition $\kappa_{11} = \kappa_{22}$	Finite element simulations confirm robust and optimized energy decay	Major advancement: universal and optimized stability under coupled control laws

Limitations:

While this study provides a robust framework for the stability of non-uniform Rayleigh beams, several physical and mathematical assumptions should be acknowledged:

- The coefficient functions $r(x)$, $I(x)$, and $EI(x)$ are assumed to be sufficiently smooth (C^4 -class) and strictly positive on $[0, 1]$. These requirements are standard for ensuring mathematical regularity and the well-posedness of the system.
- The boundary control laws are linear and applied only at the free end.
- The control coefficients $\alpha, \beta, \kappa_{11}, \kappa_{22}, \kappa_{12}, \kappa_{21}$ are non-negative, and the following dissipativity condition must be strictly satisfied to guarantee energy decay:

$$4\kappa_{12}\kappa_{21} > (\kappa_{11} + \kappa_{22})^2.$$

Although these assumptions effectively ensure the exponential stability of the system, the linear nature of the model implies a focus on small oscillations, which may limit its direct applicability to systems experiencing large deformations. Future investigations could extend this analysis to non-linear control laws to address high-amplitude vibrations or explore structures with more complex, non-homogeneous boundary conditions.

7. Conclusion

In this work, we considered a general Rayleigh beam model with variable coefficients subject to a full family of coupled boundary controls. The key contributions of this paper are as follows.

- Well-posedness of the system: we employed the well-known semigroup theory to prove that the system is well-posed.
- Exhaustive spectral analysis: using the spectral method to estimate the eigenvalues, we demonstrated that their asymptotic expansions depend exclusively on the moment control of the rotational velocity.

- Exponential stability: we established that the optimal energy decay rate is determined by the spectral abscissa. Consequently, we deduced that the system is exponentially stable if the moment control associated with velocity and the force control associated with rotational velocity are equal.
- Physical interpretation: the asymptotic expression (3.17) proves that the spectrum remains bounded away from the imaginary axis, which is an essential condition for proving exponential stability via the Riesz basis theorem. Furthermore, the requirement $\kappa_{22} = \kappa_{11}$ established in this work indicates that achieving optimal dissipation necessitates a balance between shear forces and bending moments. This finding aligns with established active control strategies used in structural engineering.
- A numerical investigation: we used the finite element method to reduce the boundary value problem to a first-order differential equation, ensuring rigorous simulations.

By integrating general boundary conditions, we have successfully stabilized a vibrating Rayleigh beam. These results extend and improve several existing findings in the study of Rayleigh models. The numerical simulations carried out have confirmed the theoretical results obtained in this paper. These results represent a significant advancement in the control of non-uniform Rayleigh beams, offering greater flexibility through multi-parameter control.

References

- [1] J.M. Wang, Riesz basis property of some infinite-dimensional control problems and its applications, Ph.D Thesis, The University of Hong Kong (2004). <https://hub.hku.hk/handle/10722/33589>
- [2] M.D. Aouragh, N. Yebary, *Stabilisation exponentielle d'une équation des poutres d'Euler-Bernoulli à coefficients variables*, Annales Mathématiques Blaise Pascal 16(2) (2009) 483–510. <http://dx.doi.org/10.5802/ambp.275>

- [3] K.K.I. Teya, G.J.M. Bomisso, K.A. Touré, A. Coulibaly, *Study of the Stability Properties for a General Shape of Damped Euler–Bernoulli Beams under Linear Boundary Conditions*, Abstract and Applied Analysis 2023 (2023) ID 9939530.
<https://doi.org/10.1155/2023/9939530>
- [4] T.K. Augustin, C. Adama, K.A.A. Hermith, A.J. Kablan, *Exponential stability of variable coefficients Rayleigh beams with indefinite damping under a force control in moment and velocity*, Far East J. Math. Sci. (FJMS) 99(7) (2016) 993–1020.
<http://dx.doi.org/10.17654/MS099070993>
- [5] B.Z. Guo, *Basis property of a Rayleigh beam with boundary stabilization*, Journal of Optimization Theory and Applications 112(3) (2002) 529–547.
<https://doi.org/10.1023/A:1017912031840>
- [6] J.M. Wang, G.Q. Xu, S.P. Yung, *Exponential stability of variable coefficients Rayleigh beams under boundary feedback control: a Riesz basis approach*, Syst. Control Lett. 51 (2004) 33–50.
<https://www.sciencedirect.com/science/article/abs/pii/S0167691103002056>
- [7] J.M. Wang, S.P. Yung, (2006) *Stability of a nonuniform Rayleigh beam with indefinite damping*, Syst. Control Letters 55 (2006) 863–870.
<https://doi.org/10.1016/j.sysconle.2006.04.002>
- [8] A.A.H. Kouassi, F.N. Diop, S.A.J. Yapi, K.A. Touré, *Stability of variable coefficients Rayleigh beams with indefinite damping under a force control*, Far East Journal of Dynamical Systems 38(1) (2025) 127–157.
<https://doi.org/10.17654/0972111825006>
- [9] W. Eberhard, G. Freiling, *Stone-reguläre Eigenwertprobleme*, Math. Z. 160 (1978) 139–161.
<https://doi.org/10.1007/BF01214265>
- [10] C. Tretter, *On fundamental systems for differential equations of Kamke type*, Math Z. 219 (1995) 609–629.
<https://doi.org/10.1007/BF02572384>
- [11] H. Brezis, *Analyse fonctionnelle, Théorie et applications*, Masson (1983).
<https://search.worldcat.org/fr/title/1023975579>
- [12] Z. Liu, S. Zheng, *Semigroups associated with dissipative system*, Research Notes in Mathematics 398, Chapman and Hall/CRC (1999).
https://books.google.ci/books/about/Semigroups_Associated_with_Dissipative_S.html?id=ReG5eHHshpoC&redir_esc=y
- [13] B. Rao, *A Compact Perturbation Method for the Boundary Stabilization of the Rayleigh Beam Equation*, Appl Math Optim 33 (1996) 253–264.
<https://link.springer.com/article/10.1007/BF01204704>
- [14] M.A. Naimark, *Linear Differential Operators*, Vol. I, Frederick Ungar Publishing Company, New York, (1967).
<https://archive.org/details/lineardifferenti0000naim>
- [15] R.F. Curtain, H.J. Zwart, *An Introduction to Infinite Dimensional Linear System Theory*, Springer, New York (1995).
<https://doi.org/10.1007/978-1-4612-4224-6>
- [16] A.P. Abro Goh, G.J.M. Bomisso, K.A. Touré, A. Coulibaly, *A numerical study for a flexible Euler–Bernoulli beam with a force control in velocity and a moment control in rotating velocity*, J. Indian Math. Soc. 90 (2023) 125–148.
<https://doi.org/10.37418/amsj.9.10.77>
- [17] A.P. Abro Goh, K.A. Touré, G.J.M. Bomisso, *Influence of control parameters on the stabilization of an Euler–Bernoulli flexible beam*, J. Numer. Anal. Approx. Theory, 53 (2024) 5–19.
<https://doi.org/10.33993/jnaat531-1384>
- [18] M. Bzeih, T. El Arwadi, M. Hindi, *Numerical analysis and simulation for Rayleigh beam equation with dynamical boundary controls*, Arab. J. Math. 10 (2021) 331–349.
<https://doi.org/10.1007/s40065-021-00310-8>
- [19] M.E. Patrice, *On the numerical approximation of the spectrum of a flexible Euler–Bernoulli beam with a force control in velocity and a moment control in rotating velocity*, Far East Journal of Mathematical Sciences (FJMS) 126(1) (2020) 13–31.
<https://doi.org/10.17654/MS126010013>
- [20] S.A.J. Yapi, A.A.H. Kouassi, F.N. Diop, K.A. Touré, *Numerical method for variable coefficients Rayleigh beams with indefinite damping under a force control in moment and velocity*, International Journal of Numerical Methods and Applications 25(2) (2025) 251–279.
<https://doi.org/10.17654/0975045225011>
- [21] M. Miletic, A. Arnold, *A piezoelectric Euler–Bernoulli beam with dynamic boundary control: Stability and dissipative FEM*, arXiv:1507.02542 [math.AP] (2015).
<http://dx.doi.org/10.1007/s10440-014-9965-1>